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The Unreasonable Effectiveness of Mathematics: From Hamming to Wigner and Back Again

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Abstract

In a paper titled, "The Unreasonable Effectiveness of Mathematics", published 20 years after Wigner's seminal paper, the mathematician Richard W. Hamming discussed what he took to be Wigner's problem of *Unreasonable Effectiveness* and offered some *partial* explanations for this phenomenon. Whether Hamming succeeds in his explanations as answers to Wigner's puzzle is addressed by other scholars in recent years (Azeri 2020) I, on the other hand, raise a more fundamental question: does Hamming succeed in raising the same question as Wigner? The answer is no. My goal is to show that Hamming's reading misses Wigner's highly original formulation of the problem.Through a close and contextual reading of Wigner's work, as I will show, we are led in new directions in addressing and solving the applicability problem.

Keywords Applicability of mathematics · Modern physics · Mathematics · Wigner · Unreasonable effectiveness · Hamming · Invariance principles

1 Introduction

In a lecture delivered at New York University in May 1959, the physicist Eugene Wigner invigorated and reintroduced the question of the applicability of mathematics, under the striking title *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*—a formulation that has influenced and shaped debates over this problem to this day. In this lecture, which was published a few months later in *The*

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Communications in Pure and Applied Mathematics (1960), Wigner described the relationship between mathematics and physics as a "miracle"or a "mystery," a phenomenon for which he asserts there is no rational explanation.[1], published in *European Journal of Philosophy*, Vol. 7, 1999. I would like to thank Akhilesh Shridar and Andre Carothers for their helpful and inspiring comments.

From the start, the lecture's title, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," boldly states the paper's conclusion. In the body of the paper, Wigner insists that:

> [T]he enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it. [23, p. 223]
> It is difficult to avoid the impression that a miracle confronts us here, quite comparable in its striking nature to the miracle that the human mind can string a thousand arguments together without getting itself into contradictions, or to the two miracles of the existence of laws of nature and of the human mind's capacity to divine them. [23, p. 229]

And finally he ends the lecture by another striking statement of this conclusion:

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning. [23, p. 237]

There is no shortage of responses to what scholars take to be Wigner's problem. (See Steiner [20], Sarukkai [19], Grattan-Guinness [8], Longo [13], Colyvan [5], Pincock [18], Velupillai [21], Lützen [14], French and Bueno [1], Islami [10] among others) One such response, from the mathematician Richard Hamming, was to do what Wigner has evaded. That is, "to give some partial answers to the implied question of the title," where the title reads, "The Unreasonable Effectiveness of Mathematics." [9, p. 83]

In my view, there is not one *single* problem under this title [12]. Instead, there are a host of epistemological, metaphysical and semantic problems. So, I will start by asking, what is Hamming's formulation of the problem of unreasonable effectiveness? Looking at a few papers makes this point a bit more clear. According to Penrose, for instance, the effectiveness of mathematics is an indicator that we live in a mathematical universe, which is a metaphysical claim [17]. Sarukkai, on the other hand, argues that mathematics is a language for the description of nature, similar to English or any other natural language, which is a more epistemological problem

[1] The title of the present paper is inspired by Jürgen Habermas's paper, "From Kant to Hegel and back again".

[19]. Frege, however, is concerned with the validity of the application of mathematics in different contexts, as a semantic problem [7].[2]

2 What is Hamming's Problem of *Unreasonable Effectiveness?*

Very early on in his paper, Hamming makes a distinction between two sides of the question,"why is mathematics so unreasonably effective?": the *material* side and the *logical* side. He understands the logical side of the question as being concerned with mathematics as our *mental tool* to carry out long chains of reasoning. And he considers that mathematics might be even so *defined*. It is in relation to this side of the question, he notes, that the consistency of mathematics plays a central role in the work on the foundations of mathematics. The logical side of the question highlights the deductive role of mathematics in our everyday reasoning as well as in the sciences. This side of the question is *not* Hamming's main focus.

On the material side, the question is different. There we ask, why is it that the world seems to admit of logical explanations? More precisely, why is it that the world seems to be organized in a logical pattern that parallels that of mathematics? Hamming writes,

> In a sense I am in the position of the early Greek philosophers who wondered about the material side, and my answers on the logical side are probably not much better than theirs were in their time. But we must begin somewhere and sometime to explain the phenomenon that the world seems to be organized in a logical pattern that parallels much of mathematics, that mathematics is the language of science and engineering. [9, p. 81]

The material side of the question is concerned with these issues: (1) the world seems to follow the logical pattern of mathematics, and (2) mathematics is the language of science and engineering. The way that Hamming puts these questions suggests that they are closely connected or even identical. He doesn't provide further justification for this claim.

In my view, these two claims are not as closely related as the quote implies. In particular, the first question is concerned with the metaphysical aspect of the effectiveness of mathematics, namely how the world is, and whether it has a mathematical structure. The second question is about the relationship of mathematics to our knowledge in natural sciences and engineering, which is mainly an epistemological question. Of course, one might argue that having a realist attitude toward scientific theories, which takes theories to mirror patterns in the world, closes the metaphysical-epistemological gap. This might be the case. However, Hamming doesn't (at least explicitly) adopt such a position nor does he give an argument in favor of

[2] For a discussion concerning different versions of the problem, see Steiner's 1998 book [20]. There, Steiner makes a distinction between what he calls the semantic problem, the descriptive problem and the role of mathematics in discovery. Moreover, see [12] for a discussion on what they call the "distinctness thesis" [12].

scientific realism. Also, it is hard to see in what way we can be realists about engineering. What seems to be the case, given how he formulates the rest of the paper, is that in his view mathematics is a language, or a good tool, fit for the purposes of engineering as well as natural sciences.[3]

Adopting an“experimentalist” approach to the question, as he describes himself, Hamming explains what he means by the unreasonable effectiveness of mathematics through a few examples in engineering.[4] His first example is that of using mathematics in the design of the atomic bomb in the Second World War. There, Hamming asks, “how was it that numbers we so patiently computed on the primitive relay computers agreed so well with what happened on the first test shot at Alamogordo?” He suggests that this is surprising that there were no experiments, even small-scale ones, that could check the computations before doing the actual test. More surprisingly, he suggests that this is not an isolated phenomenon. His second example is the use of mathematics in telephone computations in the Bell system, together with the work on traveling wave tubes, equalization of television lines, and the stability of complex communication systems. Hamming’s point is that these are only a few examples of the extensive use of mathematical manipulations with remarkable success in almost all of science and engineering.

Hamming uses these examples to show that mathematics, more precisely the *human-made* mathematics, provides a reliable model of the universe, from which we can predict what happens in the future. This mathematics tends to be simple and elementary, especially the kind that is used in engineering. In this light, the question is, “how can it be that the simple mathematics suffices to predict so much?” [9, p. 83] The idea is that simplicity is an anthropocentric standard and it appears surprising that it provides a successful guide to predictions in the world.

Further along in Hamming’s paper, it becomes more and more clear that the kind of mathematics that he finds unreasonably effective is the mathematics of elementary arithmetic, algebra and calculus. What he finds to be one of the most striking cases of the unreasonable effectiveness is the use of numbers, in particular counting numbers and elementary operations, such as addition:

> The integers seem to us to be so fundamental that we find them wherever we find intelligent life in the universe. .. Is it not remarkable that 6 sheep plus 7 sheep makes 13 sheep; that 6 stones plus 7 stones make 13 stones? Is it not a miracle that the universe is so constructed that such a simple abstraction as a number is possible. To me this is one of the strongest examples of the unreasonable effectiveness of mathematics. Indeed, I find it both strange and unexplainable. [9, p. 84]

[3] As it looks like there is not much that Hamming says by way of settling these questions, I, too, will set them aside. For an interesting discussion, see Colyvan 2001 [4].

[4] Bringing examples from engineering might partially explain why he drops the last part of Wigner’s title - The unreasonable effectiveness of mathematics in natural sciences- and talks about the effectiveness of mathematics *simpliciter*. I hope to give some other explanations for his choice of title and the way it differs from Wigner’s in the course of this paper.

As the quote suggests, it is not the more advanced mathematics of complex analysis and abstract algebra that strikes Hamming as “strange” and “unexplainable” but it is the possibility of the abstraction of natural numbers 1,2,3,.. and the simple operations such as addition between them.

Frege asked a similar question about arithmetic. He, too, was concerned with the use of mathematics in different contexts, in everyday life as well as in sciences. More clearly, he asked what account of numbers can explain their use in pure contexts such as arithmetic, consistently with their use in mixed contexts such as everyday life and natural sciences. To clarify his question, consider the following argument:[5]

1. $7 + 5 = 12$
2. There are seven apples on the table.
3. There are five pears on the table.
4. No apple is a pear.
5. Apples and pears are the only fruits on the table. Therefore,
6. There are exactly twelve fruits (or pieces of fruit, if you prefer) on the table

In this argument, the first premise belongs to arithmetic; it is in fact a theorem of arithmetic, in which the numbers seem to be used as nouns. In the second and third premises, as well as the conclusion, however, numbers are used in mixed contexts in the form of adjectives. The problem here is to find an interpretation of numbers in all these statements (arithmetical and non-arithmetical) that make this argument valid. Closely connected to this problem is a metaphysical question: how is it that numbers, as abstract objects, are used to describe physical objects- pears and apples in this example? Putting it this way, it seems that a classical problem of dualism is confronting us.

According to Frege, this is not the right way to think about the problem. In Frege’s view, it is not the case that numbers are applied to empirical objects. Rather he suggests, they are applied to empirical *concepts*. To him, numerals such as ’7’, ’5’ and so on are singular terms, proper nouns to be more precise, in all contexts. What do they refer to? Numbers (that is 7, 5,..).

The crucial point is how to understand numbers according to this view. For Frege, numbers are second-order concepts, whose extension are sets of concepts (including pears on the table, students in a class, etc).[6] He writes,

> The laws of number, therefore, are not really applicable to external things; they are not laws of nature. They are, however, applicable to judgments holding good of things in the external world: they are laws of laws of nature. [7, p. 87]

In this way the“metaphysical gap” also disappears, at least with respect to numbers and their application to empirical objects. Although his proposal is mostly focused on arithmetic, it can be extended to other parts of mathematics.

[5] I use Steiner–Dummett formulation of the problem. For a more detailed discussion see [20, pp. 15–23].

[6] Given this, numerals are names for (or nouns whose references are) second order concepts.

Although Frege's solution continues to guide us, he did not address *all* problems of the applicability of mathematics. More specifically, Hamming seems to be concerned with a different version of the problem. For him, as I understand, it is not the *validity* of the application of arithmetic in the pure and mixed contexts that appears strange and unexplainable, but it is the *usefulness* of it. In other words, in relation to the most elementary mathematical entities, namely numbers, Hamming asks, how is it that we can apply numbers to the concepts of physical objects? That is, what properties must these objects have to be countable by numbers? To put it somewhat differently, what is it about our world that makes the abstraction of numbers possible? These were not Frege's questions.

Hamming, in my view, is right to puzzle over the abstraction of numbers. It is a contingent fact about our universe that it is constituted in such a way that abstractions of this sort are possible. Were it not the case that bodies were reasonably stable, we wouldn't be able to abstract numbers and arithmetic would lose its application. To put it more precisely, objects that are important in everyday life and in science, to which arithmetic applies, remain the same under a group of translations in space and time. Using Steiner's example, the number of coins in my pocket, remains the same no matter where I am or what time of the day, month or year it is [20, p. 24]. Moreover, they remain the same, regardless of whether I put some nuts in my pocket or take out my keys. All objects that we can count "stay constant" for long enough so that we can count them. As Steiner also points out, without this reasonable stability and invariance properties, arithmetic would become useless, but more importantly "human experience would be impossible, not only arithmetic" [20, n7, p. 25].

The situation is similar in cases of addition and multiplication. For instance, what makes addition useful is that gathering or assembling objects preserves their important properties such as their stable existence. When we gather two groups of objects, as in Frege's example, we maintain their stable existence. The operation of addition is also invariant under translations in time and space. The case of multiplication is similar. (Note that it is not so striking that the arithmetical operation of addition is applicable to the act of gathering since addition is a result of abstraction from gathering, as Mill and Locke have commented.[7]) Given these, the argument goes, there is nothing unexpected and mysterious about the usefulness of counting, addition and multiplication. We can easily come up with requirements that a quantity needs to satisfy in order for it to have an "additive structure", such that we can use the operation of addition with respect to it.

[7] Steiner writes, "Consider the number of stones. We can distinguish between (a) the (scattered) physical object X of which the stones are parts; and (b) the set of Y of the stones, i.e. the abstract collection of which the stones are *members*. The physical object (a) is called the "mereological sum" of the stones. Now, it is a physical fact that the scattered physical object X retains the same stones as its parts under a wide range of physical changes, such as gathering the stones together. The parts of X which are stones, therefore, are members of the same set Y over time (thinking with the vulgar). Thus the invariant structure of the object X is the set Y. The conclusion is that in the arithmetic of sticks and stones, the concept of a mereological sum is an application of the concept of a set. This is not too surprising since the concept of set is an abstraction from that of a mereological sum" [20, n 11, p. 27].

Although these remarks might convince Hamming in the case of natural numbers and the possibility of arithmetical operations such as addition, they don't seem to go far enough. As Hamming points out, it is not easy to explain the effectiveness of the more advanced sets of numbers, from fractions to real numbers, transcendental numbers and finally complex numbers. Complex numbers[8] particularly provide a striking example of a more abstract structure that is *unreasonably* effective. When Cardano[9] introduced these numbers in sixteenth century for the purpose of solving quadratic equations, he couldn't have imagined that one day they would become useful in physics to the extent that one feels, as Hamming puts it, that "God made the universe out of complex numbers" [9, p. 85].[10]

In general, Hamming's claim is that the number system that has been developed through abstraction, generalization, increased simplicity and aesthetic reasons (all internal to mathematics) provides us with a system that "is unreasonably effective even in mathematics itself". The idea is that even though it might seem to be the case that the original sources of mathematics are forced upon us by nature, "in the development of so simple a concept as number we have made choices for the extensions that were only partly controlled by necessity and often, it seems to me, more by aesthetics". What is surprising, Hamming continues, is that in trying to "make mathematics a consistent, beautiful thing...we have had an amazing number of successful applications to the real world." The question is how can we explain this success in the application of mathematics? (Or should we convince ourselves that it is a completely unreasonable phenomenon?)

At the core of Hamming's account in his paper lies what he calls *partial explanations*:

1. We see what we look for. After all, no one should be surprised to see the whole world blueish if one has blue glasses on her eyes.

[8] By a complex number, we mean a number of the form $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$ is the imaginary unit.

[9] To be more exact, the imaginary roots of negative numbers were first defined in sixteenth century by Cardano and Bombellini to provide solutions for quadratic equations. Around the same time many mathematicians were working on the *Fundamental Theorem of Algebra* about the number of roots of polynomials with real coefficients. Cardano is one such mathematician. Introduction of complex numbers is crucial to FTA. It reads: every polynomial of degree n with real coefficients has exactly n complex roots. Prior to that mathematicians agreed that -1 (as well as all negative numbers) has no square root. Once these strange numbers appeared, they were called "sophistic," "inexplainable," "nonsensical,""absurd" or "impossible" [2, p. 10]. Later in an attempt to use these numbers, mathematicians called them "imaginary" (a word used first by Descartes) or as Leibniz put it,"existing only in mind," dismissing the prior descriptions- "nonsensical," "absurd" and "impossible." See also Magic of Complex numbers in [15] for more details.

[10] About complex numbers, Penrose writes, "While at first, it may seem that the introduction of such square roots of negative numbers was just a device- a mathematical invention designed to achieve a specific purpose- it later becomes clear that these objects are achieving far more than that for which they were originally designed...When Cardano introduced his complex numbers [in 1539] , he could have no inkling of the many magical properties which were to follow- properties which go under various names, such as Cauchy integral formula, the Riemann mapping theorem, the Lewy extension property" [16, p. 125].

2. We select the kind of mathematics to use. That is, when the mathematical tools that we have chosen are not adequate in a particular case, we choose a different kind, e.g. vectors instead of scalers.
3. Science in fact answers comparatively few problems. It of course doesn't say much about our long-asked questions about God, Justice, Beauty, Truth and so on.
4. The evolution of man has provided the model. We are able to think about objects especially the macroscopic ones, an ability that surely has had survival value [9, pp. 87–89].

All of these explanations, at first, seem to give a more or less convincing answer to Wigner's puzzle, or at least that is how they are commonly understood. However, I don't think this is quite right. To the extent that they provide answers to Wigner's puzzle, they have already been acknowledged by Wigner, in one form or another, in his 1960 paper. The rest of the paper aims to make this point clear.[11]

3 Is Hamming's Problem Wigner's Puzzle?

A quick look at Wigner's 1960 paper, "The Unreasonable Effectiveness of Mathematics in Natural Sciences", makes us realize that he devotes a large portion of the paper to the question of "What is physics?". Granted that he starts his discussion by asking "What is mathematics?" but besides a few rather sketchy remarks, he doesn't say much about the nature of mathematics. Instead, he focuses on the nature of physics, and to make a case about the effectiveness of mathematics, he cites three examples: Newtonian laws of motion, Heisenberg's formulation of quantum mechanics, and Quantum Electrodynamics or the theory of Lamb shift, all of which belong to physics. It is evident that Wigner's problem is the unreasonable effectiveness of mathematics in *modern physics*, or in post-Newtonian/Galilean physics to be more precise. And it seems to me, it is in characterizing this physics, as opposed to Aristotelean physics, other branches of natural sciences and engineering, that we start to find answers to Wigner's question of the applicability of mathematics.

In contrast, Hamming doesn't even raise a question about the nature of modern physics (at least explicitly). His paper, for the most part, is an attempt to characterize what he thinks mathematics is. Moreover, Hamming's examples are chosen mostly from engineering, e.g. the use of mathematics in the building of an atomic bomb or in a Bell system, as were mentioned earlier. At first this difference might seem to amount to nothing more than a difference in taste between Wigner and Hamming which has led one to discuss physics and the other to discuss engineering. But that's not how I see the difference and I hope to make the case in the remainder of this paper.

[11] Hamming's discussion of the role of evolution and the "thoughts that we cannot think" (in fourth explanation) seems to be very dense, vague and also interesting. For the sake of simplicity, I will keep this explanation out of my discussion.

What Wigner finds most puzzling are the abstract mathematics of complex analysis, analytic functions, Fourier transforms and so on which are developed based on formal beauty, and are increasingly used in twentieth century physics.[12] For him, elementary arithmetic and geometry don't provide a particularly worrisome case since in his view they seem to be suggested to us by nature. In the section on *What Is Mathematics?*, he writes,

> whereas it is unquestionably true that the concepts of elementary mathematics and particularly elementary geometry were formulated to describe entities which are directly suggested to us by the actual world, the same does not seem to be true of the more advanced concepts, in particular the concepts which play such an important role in physics. [23, p. 224]

From the start, Wigner seems to be agree with an explanation of the sort I discussed earlier which makes the effectiveness of counting numbers *reasonable*.

In relation to more advanced mathematical structures, what surprises Wigner is that what has been developed solely based on mathematical beauty[13] ends up playing such a fundamental role in our studies of the physical phenomena, e.g. in quantum mechanics and quantum electrodynamics. Wigner emphasizes that it is by no means the case that these concepts are simple or the simplest possible concepts that are bound to appear in physics:

> It is not, however, as is so often stated, that this had to happen because mathematics uses the simplest possible concepts and those were bound to occur in any formalism. As we saw before, the concepts of mathematics are not chosen for their conceptual simplicity- even sequences of pairs of numbers are far from being the simplest concepts- but for their amenability to manipulations and to striking, brilliant arguments. Let us not forget that the Hilbert space of quantum mechanics is the complex Hilbert space, with a Hermitian scalar product. Surely to the unpreoccupied mind, complex num-

[12] His own list includes complex numbers, algebras, linear operators, and, Borel sets and of course 'this list could be continued almost indefinably' [23, p. 224]. These are advanced concepts that were developed not with an eye on their applicability in nature but rather based on the formal beauty they manifest.

[13] What Wigner exactly means by mathematical beauty is not quite clear. However this problem is not limited to Wigner. Some physicists and mathematicians like Dirac think mathematical beauty cannot be defined any more than beauty in art can be defined. In my view, what Wigner has in mind, at least in part, is a notion of beauty that is closely connected to manipulatability and generality. New concepts such as complex numbers for instance, were invented at first as solutions to quadratic equations. So their purpose originally was to enable mathematicians to solve a wider range of equations. Later as they were linked to their geometric counterparts, namely the ordered pairs of real numbers, they proved to be an extension of the reals, whose arithmetical and algebraic operations were natural extensions of operations on the set of real numbers. Moreover, they enabled mathematicians to prove powerful theorems in different fields of mathematics, e.g. in analysis for complex functions, which didn't hold for real functions. This meant that complex numbers have a richer structure, with many properties that make them more amenable to manipulatability. Wigner, for instance, emphasizes the importance of the theory of analytic functions defined on the complex plane, which are central to quantum mechanics. A mathematician justifies her interest in the study of complex numbers, Wigner writes, by appealing to "many beautiful theorems in the theory of equations, of power series, and of analytic functions in general, which owe their origin to the introduction of complex numbers" [23, pp. 224-225]. For a discussion of analytic functions see [3].

> bers are far from natural or simple and they cannot be suggested by physical observations. [23, p. 229]

The idea is that formal beauty, and closely connected to it, manipulability and generality, are internal to the mathematical practice; it is by virtue of appealing to our aesthetic sense that the mathematician can demonstrate his ingenuity and skills. Moreover, the mathematics that is developed in this way is not conceptually simple, as we can see in the case of complex numbers and complex Hilbert spaces. What seems to be miraculous is that complex numbers, developed in this way, "are not a calculational trick of applied mathematics" but "come close to being a necessity in the formulation of the laws of quantum mechanics" [23, p. 229]. It is therefore the indispensability of such mathematical structures that strikes Wigner as *unreasonably* effective.

It is, therefore, the effectiveness of complex numbers that Wigner takes to play the role that counting numbers were playing for Hamming (as one of the strongest cases of the unreasonable effectiveness). Wigner writes:

> The complex numbers provide a particularly striking case. Certainly, nothing in our experience suggests the introduction of these quantities. Indeed, if a mathematician is asked to justify his interest in complex numbers, he will point, with some indignation, to the many beautiful theorems in the theory of equations, of power series and of analytic functions in general, which owe their origin to the introduction of complex numbers. [23, p. 225]

The postulates of quantum mechanics as formulated by Dirac and von Neumann, Wigner notes, make an extensive use of these numbers. States in quantum mechanics are vectors in a complex Hilbert space, and observables are self-adjoint (Hermitian) operators on these vectors. As I quoted before, since this complex Hilbert space, which has come close to a necessity in the formalism of quantum mechanics, is by no means simple, it is hard to "avoid the impression that a miracle confronts us here" [23, p. 229].

Let's remind ourselves, once again, that Hamming was mostly concerned with elementary branches of mathematics that are employed in engineerings. There he asks, "how can it be that simple mathematics being after all a product of the human mind, can be so remarkably useful in so many widely different situations?" [9, p. 83] In a different place, he notes, "how can it be that simple mathematics suffices to predict so much?" Simplicity, of course, plays a significant role in the kind of mathematics we use in engineering, whereas it doesn't have a similar importance in physics.

In his 1939 paper titled "The Relation between Physics and Mathematics" Dirac sheds more light on this problem. There he argues that it was Einstein who changed the trend in physics. Dirac states that it was the discovery of theory of relativity that made it necessary to modify the principle of simplicity. Since Einstein thought an elaborate technique needs to be developed before the simple law of gravitation could be written, the principle of simplicity had to be given up in favor of mathematical beauty; that is, simplicity became subordinate to beauty.

Dirac writes, "The theory of relativity introduced mathematical beauty to an unprecedented extent into the description of Nature" [6, p. 123].

About the structure of mathematics as a whole, also, Hamming and Wigner seem to differ. Wigner advocates a view of mathematics wherein it is understood as an accumulative discipline in which the newly introduced concepts, such as complex numbers, add to the already established set of concepts and theorems. Hamming, on the other hand, argues that mathematics is constantly changing in such a way that the introduction of new concepts leads to the falsity of the proofs of many theorems. He even makes an anecdotal point quoting the ex-editor of *Mathematical Reviews* claiming that half of the new theorems published in that journal are true although their proofs are false. He writes,

> [W]e see that one of the main strands of mathematics is the extension, the generalization, the abstraction... of well-known concepts to new situations. But note that in the very process the definitions themselves are subtly altered. Therefore, what is not so widely recognized, old proofs to theorems may become false proofs. The old proofs no longer cover the newly defined things. [9, pp. 85–86]

Not only are definitions subtly altered in Hamming's view, but also postulates undergo change in such a way that the postulates we find true today might be pronounced false in the future, as the bold words in his paper read: "The postulates of mathematics were not on the stone tablets that Moses brought down from Mt. Sinai."

The process of defining a new concept, according to Hamming, is along the following lines: we start from a vague concept, and create different sets of postulates to define this concept. Among these sets, we finally settle on one but in doing so we modify and sharpen the initial concept. Then, based on, these postulates, we (generate and) prove theorems which lead to a more "beautiful and consistent mathematics". In that process, given the kind of theorems we want to prove, we might change the postulates or choose a completely different set. And that is very plausible, Hamming claims, given that it is usually the case that postulates come from theorems and not the other way around (Euclid's postulates, for instance, came from the Pythagorean theorem, not vice versa.). By way of summarizing this account, Hamming states, "mathematics has been made by man and therefore is apt to be altered rather continuously by him" [9, pp. 86–87].

With these differences in mind, I will go back to the question I asked before: why is it that Wigner puts so much emphasis on modern physics? Also why is it that Hamming, instead, focuses on mathematics?

The question of the unreasonable effectiveness of mathematics for Wigner is closely connected to the nature of modern physics. What demarcates Newton from most physicists prior to him, is the realization that in order to formulate laws of nature, one has to distinguish between initial conditions, and laws on nature. Initial conditions are responsible for "accidental" aspects of the world and are not relevant to the result of the experiment. Laws of nature are the statements of regularities, on the other hand.[14] Kepler Wigner notes, didn't have this understanding; so at the same time that he was formulating his laws of planetary motion, he "tried to explain the size of the planetary orbits and their periods." Wigner writes:

> The surprising discovery of Newton's age is just the clear separation of laws of nature on the one hand and initial conditions on the other. The former are precise beyond anything reasonable; we know virtually nothing about the latter. [25, p. 40]

However, separation of initial conditions doesn't by itself make the formulation of laws of nature possible. What is required, moreover, is the assumption that with the same initial conditions the result will be the same—that is, the laws of nature will hold under different circumstances. So that we can assume the result of the experiments done in lab or carried out on a small sample of observations holds universally. The underlying idea is that there is nothing peculiar about the time in which we make the observations, nor about the location in which the experiments are being done. This is what gives laws their universal character. Put more mathematically, the assumption is that the laws remain the same under specified groups of transformation. The homogeneity of space and the uniformity of time have been recognized as prerequisites for formulation of laws of nature since the time of Newton in its mature sense and also in a more implicit form by Galileo.

Wigner's idea is that the world is complicated and much about it is unknown.[15] In this enormous complexity, the physicist has devised "an artifice" that makes the discovery of regularities possible. He writes:

> Man has devised an artifice which permits the complicated nature of the world to be blamed on something which is called accidental and thus permits him to abstract a domain in which simple laws can be found. These complications are called initial conditions; the domain of regularities, laws of nature. Unnatural as such a division of the world's structure might appear from a very detached point of view, and probable though it is that the possibility of such a division has its own limits, the underlying abstraction is probably one of the most fruitful ones that human mind has made. It has made the natural sciences possible. [22, p. 3]

Using a Kantian language, abstraction of laws from initial conditions is a necessary condition for the possibility of the formulation of laws of nature. The assumption of invariance principles, especially the invariance with respect to displacements in

[14] The case can be made that assuming the possibility that the system can be studied in isolation from the rest of the world, as a closed system, without disturbing the result of its study, is a necessity in any 'causal analysis' and Newton fully underwood this requirement.

[15] Given this, it is unreasonable to draw metaphysical conclusions about the 'mathematical structure' of inanimate nature from the effectiveness of mathematics in physics. Laws of nature, which are, as opposed to initial conditions, mathematizable, make up only a small portion of our physical knowledge. In the rest mathematics plays no role.

space and time, which give laws their generality and testability is another one of the necessary conditions for the possibility of the formulation of laws of nature.[16]

Invariance principles not only give laws their fundamental characteristics but also they make it possible for laws to be mathematizable. Put somewhat differently, mathematics is effective in the formulation of laws of nature in physics because laws have such invariance properties. Mathematics understood in this way, then, is the study of concepts and operations that have invariance properties. This aspect of mathematics manifests itself mostly in the study of abstract algebra as well as linear algebra, both of which play fundamental roles in twentieth century physics.[17] We can make a case that mathematical beauty also, as a criterion that is tied to generality and extension, is also closely connected to invariance properties.[18]

Wigner takes the purpose of above remarks to remind us that mathematics is useful only in a very small part of our knowledge of inanimate nature, namely in the formulation of laws of nature. And it is significant to remember that laws of nature are conditional statements "which permit prediction of some future events on the basis of the knowledge of the present" only under the exceptional circumstance "when all the relevant determinants of the present state of the world are known" [23, p. 227]. Thinking about the effectiveness of mathematics in physics, therefore, leads us to a better understanding of the structure of our physics: its hierarchical structure of events, laws of nature, and invariance principles as Wigner has proposed.[19] This

[16] Wigner writes, "The possibility of isolating the relevant initial conditions would not in itself make possible the discovery of laws of nature. It is rather also essential that, given the same initial conditions, the result will be the same no matter where and when we realize them. .. The statement that absolute time and position are never essential conditions is the first and perhaps most important theorem of invariance in physics. If it were not for it, it might have been impossible for us to discover laws of physics" [22, p. 4]. Given these remarks, it might be more accurate to say that these two abstractions jointly make a condition of the possibility of the formulation of laws of nature.

[17] Note that as I discussed before, it is also the case that simple arithmetical operations such as addition and multiplication are useful only because objects in the physical world are stable enough to be counted, and gathering for instance (from which we have abstracted the operation of addition) has important invariance properties. See Steiner [20] for a more detailed discussion.

[18] However, we need to realize that from Wigner's point of view, there is a distinction between the use of mathematics in the formulation of laws of nature, what he calls empirical law of epistemology, and the assumption of invariance principles. It is the latter that plays the role of a constitutive element in physics whereas for Wigner it is the former that gives physicists emotional encouragement to use mathematics in the formulation of laws, as he puts it, it is more "an article of faith on the part of the theorist". We hope this continues to happen in the future, as it has in the past three centuries, but it doesn't have to be the case. See [23, p. 233].

[19] Wigner writes, "There is a strange hierarchy in our knowledge of the world around us. Every moment brings surprises and unforeseeable events... truly the future is uncertain. There is, nevertheless, a structure in the events around us, that is, correlations between the events of which we take cognizance. It is this structure, these correlations, which science wishes to discover, or at least the precise and sharply defined correlations. ... We know many laws of nature and we hope and expect to discover more. Nobody can foresee the next such law that will be discovered. Nevertheless, there is a structure in the laws of nature which we call the laws of invariance. This structure is so far-reaching in some cases that laws of nature were guessed on the basis of the postulate that they fit into the invariance structure. ... This then, the progression from events to laws of nature, and from laws of nature to symmetry or invariance principles, is what I meant by the hierarchy of our knowledge of the world around us" [24, pp. 28–29].

insight into the structure of modern physics, provides some *partial* explanation for the effectiveness of mathematics, to use Hamming's language.

4 Does Hamming Fail to See Wigner's *partial* Explanation?

Although Hamming might seem to have missed Wigner's discussion of invariance principles, which arguably provides a missing link in the connection of modern physics and mathematics, a closer look shows that this is not quite right. It is not a coincidence, in my view, that right after introducing Wigner's paper, Hamming goes on to say that, "[t]he fundamental role of *invariance* is stressed by Wigner." He continues:

> It is basic to much of mathematics as well as to science. It was the lack of invariance of Newton's equations (the need for the absolute frame of reference) that drove Lorentz, Fitzgerald, Poincaré, and Einstein to the special theory of relativity.

Hamming also realizes that we can explain the use of mathematical functions such as trigonometric functions in Ptolemy's astronomy as well as in other instances based on the fact that they are time-invariant.

Later in his first partial explanation of unreasonable effectiveness, he elaborates much further on this point. There he asks why it is that we do all the analysis in terms of Fourier integrals? Of course they provide a tool for representing quantities under study, but Hamming's question is that "why are they natural tools for the problem?" [9, p. 88] The answer once again is based on their invariance properties. The idea is that the eigenfunctions of translation are complex exponential functions and we are led to these functions if we want our quantities to be time-invariant. So the assumption of time-invariance leads to complex exponentials. Moreover, we require our functions to be linear from which we are led to these eigenfunctions.

Now the question is should we think Hamming is only repeating what Wigner has already suggested? That is, can we claim that Hamming following Wigner explains the effectiveness of mathematics in modern physics on the basis of invariance principles? The answer once again is negative given the difference in their respective views of physics.

Physics for Wigner is an empirical science, in which either we start from collected data in case of Newtonian mechanics and Galilean laws of falling bodies, as his first example indicates, or we use experiments to modify and confirm our guessed theories, in the case of Heisenberg's matrix mechanics, for instance.[20] Similarly invariance principles, on Wigner's view, in particular the ones that are used in Newtonian mechanics, quantum mechanics and quantum electrodynamics (which are the examples he mentions in his 1960 paper), have their root in experience: they belong to a group that he calls *older* or *geometrical* principles of invariance. From this group, I have so far mentioned invariance with respect to displacements in space

[20] Wigner discusses these cases in his 1960 paper, see [23, pp. 230–233].

and time. The rest of this group include equivalence of all directions in space and the independence of the laws of nature from the state of motion as long as it is uniform. Although they are already present in Newtonian mechanics, it was Poincaré who clarified the formulation of these principles.[21] To put it differently, it is a contingent fact about the world we live in that some invariances hold, to an approximate degree. So the assumption of invariances, at least in part, is justified by experience.[22]

To summarize, according to Wigner, physics is the study of laws of inanimate nature which are formulated in the language of mathematics. Nevertheless, there is a close connection between the laws and empirical data. Ultimately it is the empirical tests or experiments that confirm, modify or even falsify parts of physics (including laws of nature among which invariance principles hold).

Hamming advocates a completely different view of physics. In his first *partial* explanation, following Eddington, he argues that the whole of physics or at least "a surprising amount of it" can be derived deductively by armchair thinking only.[23] It is in this way that physics seems to be almost an *a priori* endeavor, much entangled and similar to mathematics. Hamming knows that this is not a widely-held view and to make his point he brings some examples as evidence. I will mention a few.

The first example is that of Pythagoras. His suggestion is that Pythagoras is the first great physicist. His *discovery* was that we live in L_2, where the sum of squares of the two sides of a right triangle is equal to the square of the hypotenuse. Based on this discovery, Euclid's postulates were formed. As I discussed before, in Hamming's view, it is not the postulates that lead to theorems, it is the other way around.

The second example is the case of Galileo. Hamming thinks he arrived at his laws of falling bodies not based on any experiment but based on scholastic reasoning. Hamming boldly claims, "I know that the textbooks often present the falling body law as an experimental observation; I am thinking it is logical law, a consequence of how we tend to think" [9, p. 87].[24] According to Hamming, what led Galileo to think

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21 That is why Wigner calls them Poincaré group. As we know, it is not until Einstein that these principles find their importance in the theory of spacial relativity.

22 Note that invariance principles are assumed as exact statements in mathematics, but they hold approximately in physics and this leads to approximations and idealizations when we use mathematics.

23 Hamming writes, "Eddington went further than this; he claimed that a sufficiently wise mind could deduce all of physics. I am only suggesting that a surprising amount can be so deduced" [9, pp. 88–89].

24 Compare this with Wigner's discussion of the *discovery* of the laws of falling bodies. Wigner tells a story along the following lines. In seventeenth century, the laws of falling bodies were established as a result of a few observations done mostly in Italy. Based on those observations, Galileo realized that the rate of the acceleration of objects remains constant during a fall (whereas their velocity increases). He also recognized that the length of the time that it takes an object to fall from a given height is independent of its size, shape and the material out of which it is made. Galileo of course didn't understand the nature of the attractive force and that it is this force which acts on falling bodies independent of their shape, size and material. Nevertheless he succeeded in distinguishing acceleration from velocity and the independence of the rate of acceleration from these qualities of the object. As we know today, and Wigner notes, Galileo's law of falling bodies and later Newton's laws of motion were based on observations that couldn't possibly be accurate in the sense we understand accuracy today due to the effects of air resistance and the impossibility of measuring short intervals of time. Nonetheless, such "scanty" observations, as Wigner describes, led these Italian scientists to an understanding of the behavior of falling bodies and the way they travel through atmosphere, which in turn provided a basis for Newton's discovery of laws of motion. See Wigner's discussion for a more detailed account [23, pp. 225–228].

that all objects fall with the same speed regardless of their shape, volume and mass was a thought experiment. Imagine that in falling the body breaks into two pieces. Then suppose that one piece starts touching the other. We ask, "Would they now be one piece and both speed up?" What if we tie them together. Again, "How tightly must I do it to make them one piece? A light string? A rope? Glue? When are the two pieces one?" Galileo then thought, as Hamming imagines, that the body cannot possibly know how heavy it is , or whether it is one piece or many. Therefore, all bodies must fall with the same speed.[25]

The case of Newton is similar, Hamming claims. Newton's assumption was that we live in a three-dimensional Euclidean space in which energy is conserved. Once we have these two assumptions in place, the inverse square law deductively follows. The inverse square law is not the result of experiments as it is often claimed, it is just the consequence of these plausible assumptions.[26]

Hamming takes the preceding discussion of the deductive nature of physics to provide the first answer to the question of unreasonable effectiveness.[27] This not only sets his question apart from that of Wigner but also explains why his answers might not be attractive to someone who is interested in the relation of a study of nature in physics and *a priori* principles of mathematics.

The discussion of invariance principles sheds new light on the question of applicability of mathematics. By studying the nature of modern physics, as Wigner did throughout his career [10], we find answers to this question: what is it about modern physics that allows mathematics to play such a fundamental role? This is particularly illuminating when we compare the role of mathematics in physics to that of biology and other sciences [11].

5 Conclusion

Wigner's applicability problem is the problem of explaining the effectiveness of mathematics in physics. On the most common reading, Wigner's response to this question is a failure at best. As a result of failing to find any solution to the applicability problem, Wigner, on this reading, ends up calling the applicability of mathematics a *miracle* which we neither understand nor deserve. "Deserving" aside, the scholarship has focused on whether we don't really "understand" this phenomenon and therefore are justified to call it a "miracle".

[25] Hamming claims that he is not alone in thinking this way. Polya, in his *Mathematical Methods in Science*, also argues that Galileo found his law by simple reasoning and not through experiments [9, p. 87].

[26] Hamming continues with arguing that the same is true of quantum mechanics, and the use of Fourier integrals in Digital filters and distribution of physical constants. Since I believe the cases I mentioned make the point clear enough, I will not discuss these later cases. See [9, p. 88].

[27] Hamming, of course is not the only one who thinks of physics as closely resembling mathematics in its deductive, non-empirical structure. Dirac also comments that theoretical physics is, in the end, a search for beautiful mathematics. The idea is that theoretical physics and pure mathematics are moving toward unification [6].

Most of responses, I claim, belong to two groups. The first group, more or less, fails to accurately characterize Wigner's *question*, and his version of the applicability problem. Members of this group, although attempting to help Wigner by offering a solution to this problem, miss the point from the start. Hamming's 1980 paper "The Unreasonable Effectiveness of Mathematics" is a representative of this group. He, unlike many other scholars including Sarukkai [19], Grattan-Guiness [8], Velupillai [21], is sympathetic with Wigner and doesn't accuse him of posing a pseudo-problem. The difficulty with his account, then, is that what he provides a solution for is not really Wigner's problem.

The second group, just like the first, reads the conclusion of Wigner's paper from its title: that is, the effectiveness of mathematics in the natural sciences is unreasonable. While unlike the first group the members of the second group succeed in characterizing Winger's *question*, they fail to offer an accurate reading of his *answer* (his solution to the applicability problem). The project of this group then is to turn "Wigner's miracle" into a reasonable phenomenon.[28]

This paper was an attempt to defend Wigner's original formulation of the problem by defending him against the first group. I argued that the differences between Hamming's view and Wigner's, amount to different pictures of the nature of physics, mathematics and how they relate to one another. These differences lead to such incompatible world-views that their respective formulations of the "unreasonable effectiveness" turn out to have different significance and force.

Moreover, by studying the question of how mathematics is used in physics (prior to, why), we begin to illuminate the nature of modern physics and the unique role that invariance principles play. In this way, instead of starting from ready-made philosophies (of mathematics and physics), we are inevitably led to formulating our philosophical accounts based on details and case studies. This leads in new directions in formulating and solving the applicability problem by moving the focus to the details of the evolving relationship between mathematics and empirical sciences studied in diachronic and synchronic ways [12].

Hamming's explanations—in particular the first one—seem to be partially successful explanations of *his* problem of the unreasonable effectiveness of mathematics (although to my surprise he doesn't acknowledge this in his conclusion). He writes, "From all of this, I am forced to conclude both that mathematics is unreasonably effective and that all of the explanations I have given when added together simply are not enough to explain what I set to account for" [9, p. 90]. The only way that I can make sense of these remarks in the context of the whole paper, is that his modesty doesn't allow him to credit his own work. Also since this was a speech given to a mathematical audience at the meeting of the Northern California Section of MAA, I take it that he wants to persuade others in continuing the project. As he says, "I think we, meaning

[28] Mark Steiner's detailed treatment of the applicability problem in his 1998 book, *The Applicability of Mathematics as a Philosophical Problem* is a representative of this group. For Steiner, the effectiveness of mathematics in our study of modern physics provides us with evidence for the conclusion that we live in a user-friendly world. This world, as it stands, is created for us, and therefore we have sufficient reason to believe in anthropocentrism, according to which the human mind has a special place in the universe. The difficulty with this account is that it completely ignores Wigner's solution to the applicability problem. This argument needs a space of its own.

you, must continue to try to explain why ... mathematics is the proper tool for exploring the universe." I hope I am at least partially right in attributing these to his modesty.

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